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A FORMULATION OF THE BOUNDARY ELEMENT METHOD FOR AXISYMMETRIC TRANSIENT HEAT CONDUCTION

L. C. WROBEL and C. A. BREBBIA
Department of Civil Engineering, University of Southampton, Southampton SO9 5NH, U.K.

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Abstract — The present paper develops a formulation of the boundary element method for the analysis of axisymmetric transient heat conduction problems. The axisymmetric time-dependent fundamental solution is obtained by directly integrating the three-dimensional one. Due to its complexity, series expansions have to be introduced in order to make possible the analytical evaluation of the time integrals that appear in the formulation. Several results of numerical analyses are presented, including problems with time-dependent boundary conditions, and they demonstrate the feasibility of using boundary elements in space and time to solve axisymmetric heat conduction problems.

NOMENCLATURE

T,	temperature;
t,	time;
q,	flux;
K,	thermal conductivity;
ρ,	density;
c,	specific heat;
k,	thermal diffusivity;
h,	heat transfer coefficient;
T_s,	temperature of the surrounding medium;
T_0,	initial temperature;
Ω,	volume of the solid;
Γ,	boundary surface of the solid;
A,	generating area of the solid;
S,	generating boundary contour of the solid;
$r_{,n}, z_{,n}$,	direction cosines of the outward normal n to boundary S;
x_i, y_i, z_i,	Cartesian coordinates of source point;
x, y, z,	Cartesian coordinates of reference point;
r_i, θ_i, z_i,	cylindrical coordinates of source point;
r, θ, z,	cylindrical coordinates of reference point;
ϕ, ψ, γ,	interpolation functions;
w,	weighting factor;
J,	Jacobian;
δ_{ij},	Kronecker delta;
$I_0(x)$,	modified Bessel function of the first kind of order zero;
$I_1(x)$,	modified Bessel function of the first kind of order one;
$E_1(x)$,	exponential integral;
$\Gamma(\alpha, x)$,	incomplete gamma function;
N,	number of boundary nodes;
M,	number of cells;
L,	number of integration points in each cell;
P,	number of internal nodes;
V,	$N + P$.

INTRODUCTION

THE BOUNDARY element method has been attracting growing interest of engineers and mathematicians, as can be seen by the number of recently published books and conferences on the subject [1–9]. The main advantage of the method is the reduction by one of the dimensionality of the problem under consideration. Thus, considerable savings in the data input and computer CPU time required to run it can be achieved.

For nonlinear and transient problems, this advantage is partially lost since an integration over the domain is at present used [10, 11] but the method still retains its accuracy and facility of dealing with infinite regions and problems with high gradients. Furthermore, the cells employed on the domain discretization can be larger than usual finite elements and for transient problems, the use of fundamental solutions that are space- and time-dependents avoid the need of integrating step by step on time using a finite difference type scheme.

The present paper develops a formulation of the method for axisymmetric transient heat conduction. The axisymmetric fundamental solution is obtained by directly integrating the three-dimensional one [12]. In order to perform the time integrals analytically, series expansions are introduced but in general the series converge very quickly and despite the complexity of the fundamental solution, the computational effort involved in not great.

Several applications are discussed, including problems with 'radiation' and time-dependent boundary conditions. The numerical results and the small computer CPU times reported confirm the validity of the formulation.

BOUNDARY INTEGRAL EQUATION

The governing equation for three-dimensional transient heat conduction in a homogeneous, isotropic solid body, in the absence of heat generators inside the domain, is

$$K\nabla^2 T = \rho c \frac{\partial T}{\partial t} \quad \text{in } \Omega \tag{1}$$

with boundary conditions of the following types

$$T = \bar{T} \quad \text{on } \Gamma_1 \tag{2a}$$

$$q = -K\frac{\partial T}{\partial n} = \bar{q} \quad \text{on } \Gamma_2 \tag{2b}$$

$$q = h(T - T_s) \quad \text{on } \Gamma_3. \tag{2c}$$

Equation (2c) represents the convection or 'radiation' boundary condition. Also, initial conditions of the type

$$T = T_0 \quad \text{in } \Omega \tag{3}$$

need to be defined at $t = 0$.

For our numerical solution, T will be approximated and we can minimize the error thus introduced by weighting the governing equation and boundary conditions by a new function T^*. As the problem is time-dependent, we shall also have to weight the equations with respect to time. This yields the following weighted residual statement

$$\int_0^\tau \int_\Omega \left(K\nabla^2 T - \rho c\frac{\partial T}{\partial t}\right) T^* \, d\Omega \, dt$$
$$= \int_0^\tau \int_{\Gamma_1} (T - \bar{T}) q^* \, d\Gamma \, dt$$
$$- \int_0^\tau \int_{\Gamma_2} (q - \bar{q}) T^* \, d\Gamma \, dt$$
$$- \int_0^\tau \int_{\Gamma_3} [q - h(T - T_s)] \, T^* \, d\Gamma \, dt \tag{4}$$

where $q^* = -K(\partial T^*/\partial n)$.

Integrating by parts the Laplacian with respect to x_k gives

$$- \int_0^\tau \int_\Omega K \frac{\partial T}{\partial x_k}\frac{\partial T^*}{\partial x_k} \, d\Omega \, dt$$
$$- \int_0^\tau \int_\Omega \rho c \frac{\partial T}{\partial t} T^* \, d\Omega \, dt$$
$$= \int_0^\tau \int_{\Gamma_1} (T - \bar{T}) q^* \, d\Gamma \, dt$$
$$+ \int_0^\tau \int_{\Gamma_1} q T^* \, d\Gamma \, dt + \int_0^\tau \int_{\Gamma_2} \bar{q} T^* \, d\Gamma \, dt$$
$$+ \int_0^\tau \int_{\Gamma_3} h(T - T_s) T^* \, d\Gamma \, dt \tag{5}$$

where $k = 1, 2, 3$ ($x_1 = x$, etc.) and Einstein's summation convention is implied. Integrating by parts once more

$$\int_0^\tau \int_\Omega K\nabla^2 T^* T \, d\Omega \, dt - \int_0^\tau \int_\Omega \rho c \frac{\partial T}{\partial t} T^* \, d\Omega \, dt$$
$$= - \int_0^\tau \int_{\Gamma_2 + \Gamma_3} T q^* \, d\Gamma \, dt - \int_0^\tau \int_{\Gamma_1} \bar{T} q^* \, d\Gamma \, dt$$
$$+ \int_0^\tau \int_{\Gamma_1} q T^* \, d\Gamma \, dt$$
$$+ \int_0^\tau \int_{\Gamma_2} \bar{q} T^* \, d\Gamma \, dt$$
$$+ \int_0^\tau \int_{\Gamma_3} h(T - T_s) T^* \, d\Gamma \, dt. \tag{6}$$

Integrating by parts the time derivative finally gives

$$\int_0^\tau \int_\Omega \left(K\nabla^2 T^* + \rho c \frac{\partial T^*}{\partial t}\right) T \, d\Omega \, dt$$
$$- \rho c \left[\int_\Omega T^* T \, d\Omega\right]_{t=0}^{t=\tau}$$
$$+ \int_0^\tau \int_\Gamma q^* T \, d\Gamma \, dt = \int_0^\tau \int_\Gamma T^* q \, d\Gamma \, dt. \tag{7}$$

The fundamental solution to this equation, corresponding to a concentrated heat source applied at a point i is [13]

$$T^* = \frac{1}{[4\pi k(\tau - t)]^{3/2}} \exp\left[-\frac{R^2}{4k(\tau - t)}\right] \tag{8}$$

where

$$R = [(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2]^{1/2}$$

is the distance from the point of application of the concentrated heat source to the point under consideration. The fundamental solution possesses the following properties

$$K\nabla^2 T^* + \rho c \frac{\partial T^*}{\partial t} = 0 \quad \text{in } \Omega \text{ for all } t < \tau. \tag{9}$$

$$\int_\Omega T T^* \, d\Omega = T_i \quad \text{for } t = \tau. \tag{10}$$

In order to investigate the singularity that occurs in the integrals in equation (7) at time $t = \tau$, we may subtract to the upper limit of the integrals an arbitrarily small quantity ε, to avoid ending the integrations exactly at the peak of a Dirac delta function. Thus the first integral on the LHS is identically zero because of equation (9). Taking the limit as $\varepsilon \to 0$ and accounting for condition (10), equation (7) yields

$$T_i + \frac{k}{K}\int_0^\tau \int_\Gamma T^* q \, d\Gamma \, dt$$
$$= \frac{k}{K}\int_0^\tau \int_\Gamma q^* T \, d\Gamma \, dt + \left[\int_\Omega T^* T \, d\Omega\right]_{t=0} \tag{11}$$

where $k = K/\rho c$.

Equation (11) is valid for any point inside the domain but in order to obtain a boundary integral equation we have to take point i to the boundary. On doing so, one must consider the nature of the singularity of the integral in q^*, which has a discontinuity as i approaches the boundary. This gives

$$c_i T_i + \frac{k}{K}\int_0^\tau \int_\Gamma T^* q \, d\Gamma \, dt$$

$$= \frac{k}{K}\int_0^\tau \int_\Gamma q^* T \,d\Gamma\, dt + \left[\int_\Omega T^* T \,d\Omega\right]_{t=0} \quad (12)$$

where the c_i coefficient is a function of the solid angle of the boundary at point i [2] and the integral in q^* is evaluated in the Cauchy principal value sense.

For axisymmetric problems, all quantities are independent of the circumferential location, thus one integration can be performed in advance in equation (12). This is equivalent to using ring heat sources as fundamental solutions. Writing the three-dimensional solution (8) in cylindrical coordinates and integrating over a ring of radius r_i at the plane z_i, we have

$$T^* = \frac{1}{[4\pi k(\tau - t)]^{3/2}} \exp\left[-\frac{r^2 + r_i^2 + (z - z_i)^2}{4k(\tau - t)}\right] \times \int_0^{2\pi} \exp\left[\frac{r r_i \cos(\theta - \theta_i)}{2k(\tau - t)}\right] d\theta_i \quad (13)$$

which gives [14]

$$T^* = \frac{2\pi}{[4\pi k(\tau - t)]^{3/2}} \times \exp\left[-\frac{r^2 + r_i^2 + (z - z_i)^2}{4k(\tau - t)}\right] I_0\left[\frac{r r_i}{2k(\tau - t)}\right]. \quad (14)$$

Note that as $r_i \to 0$ this fundamental solution tends to the three-dimensional one. Differentiating expression (14) gives

$$q^* = -K\frac{\partial T^*}{\partial n} = \frac{K}{8\sqrt{\pi}[k(\tau - t)]^{5/2}} \times \exp\left[-\frac{r^2 + r_i^2 + (z - z_i)^2}{4k(\tau - t)}\right] \times \left\{\left(r I_0\left[\frac{r r_i}{2k(\tau - t)}\right] - r_i I_1\left[\frac{r r_i}{2k(\tau - t)}\right]\right) r_{,n} + (z - z_i) I_0\left[\frac{r r_i}{2k(\tau - t)}\right] z_{,n}\right\}. \quad (15)$$

The boundary integral equation (12) now becomes

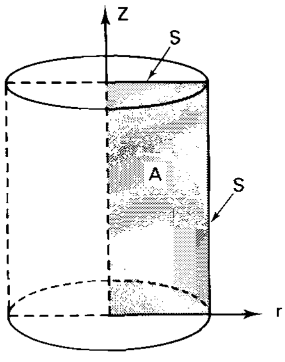

FIG. 1. Generating area and boundary contour of solid of revolution.

$$c_i T_i + \frac{k}{K}\int_0^\tau \int_S T^* q r \,dS\, dt = \frac{k}{K}\int_0^\tau \int_S q^* T r \,dS\, dt + \left[\int_A T^* T r \,dA\right]_{t=0} \quad (16)$$

where A and S are the projections of Ω and Γ, respectively, in the r^+z semiplane (Fig. 1).

NUMERICAL FORMULATION

For the numerical solution of equation (16) the boundary is discretized into a series of elements over which the geometry, temperature and flux vary according to chosen interpolation functions. One also needs to assume a certain variation on time for T and q. As these functions vary slower than T^* and q^*, it is a reasonable approximation to assume that they are constant over small intervals of time and perform the time integrations stepwise. This assumption makes possible the analytical evaluation of the time integrals in (16). After an appropriate change of variables, the LHS integral becomes

$$\int_{t_1}^{t_2} T^* \,dt = \frac{1}{2k\sqrt{(\pi d)}}\int_C^\infty I_0(2ax) x^{-1/2} e^{-x} \,dx \quad (17)$$

where

$$d = r^2 + r_i^2 + (z - z_i)^2; \quad a = \frac{r r_i}{d}$$
$$x = \frac{d}{4k(t_2 - t)}; \quad C = \frac{d}{4k(t_2 - t_1)}. \quad (18)$$

Expanding the Bessel function in series as [15, p. 375]

$$I_0(2ax) = \sum_{n=0}^{\infty} \frac{(ax)^{2n}}{n!^2} \quad (19)$$

the integral becomes

$$\int_{t_1}^{t_2} T^* \,dt = \frac{1}{2k\sqrt{(\pi d)}} \sum_{n=0}^{\infty} \frac{a^{2n}}{n!^2} \Gamma(2n + \tfrac{1}{2}, C). \quad (20)$$

For the RHS time integral in (16), we have

$$\int_{t_1}^{t_2} q^* \,dt = \frac{K}{kd\sqrt{(\pi d)}}\left\{[r r_{,n} + (z - z_i) z_{,n}] \times \int_C^\infty I_0(2ax) x^{1/2} e^{-x} \,dx - r_i r_{,n} \int_C^\infty I_1(2ax) x^{1/2} e^{-x} \,dx\right\}. \quad (21)$$

The Bessel function I_1 can be expanded as [15, p. 375]

$$I_1(2ax) = \sum_{n=0}^{\infty} \frac{(ax)^{2n+1}}{n!^2 (n + 1)} \quad (22)$$

which gives for the integral

$$\int_{t_1}^{t_2} q^* \, dt = \frac{K}{kd\sqrt{(\pi d)}} \left\{ [rr_{,n} + (z - z_i)z_{,n}] \times \sum_{n=0}^{\infty} \frac{a^{2n}}{n!^2} \Gamma(2n + \tfrac{3}{2}, C) - r_i r_{,n} \sum_{n=0}^{\infty} \frac{a^{2n+1}}{n!^2(n+1)} \Gamma(2n + \tfrac{5}{2}, C) \right\}. \quad (23)$$

All the incomplete gamma functions that appear on the above series can be evaluated in terms of $\Gamma(\frac{1}{2}, C)$ by using the following recurrence relation [16, p. 942]

$$\Gamma(n+1, C) = n\Gamma(n, C) + C^n e^{-C} \quad (24)$$

and for computational purposes we can evaluate $\Gamma(\frac{1}{2}, C)$ using a rational approximation that takes into account its relation with the complementary error function [15, pp. 262 and 299]

$$\Gamma(\tfrac{1}{2}, C) = \sqrt{\pi}\,\mathrm{erfc}(\sqrt{C}) = \sqrt{\pi}(0.3480242 - 0.0958798p + 0.7478556p^2)p\,e^{-C}, \qquad p = \frac{1}{1 + 0.47047\sqrt{C}} \quad (25)$$

with an accuracy that is sufficient for our calculations.

From (18) one notices that the value of the constant a varies between 0 (for r or $r_i = 0$ or for $d \to \infty$) and 0.5 (for $r = r_i$, $z = z_i$). All the series that appear on expressions (20) and (23) converge very quickly for small values of a but slowly as $a \to 0.5$. In fact, they do not converge for $a = 0.5$, due to the singularity at $r = r_i, z = z_i$. So, from the computational point of view, it is not convenient to use expansions (19) and (22) for values of a in the vicinity of $a = 0.5$.

In order to overcome this problem, we can use asymptotic expansions of the Bessel functions that are valid for large values of their arguments. Thus, whenever x is large we can write [15, p. 377].

$$I_0(2ax) = \frac{e^{2ax}}{2\sqrt{(\pi a x)}} \left[1 + \sum_{n=1}^{\infty} \frac{f_1(n)}{n!(16ax)^n} \right] \quad (26)$$

$$I_1(2ax) = \frac{e^{2ax}}{2\sqrt{(\pi a x)}} \left[1 + \sum_{n=1}^{\infty} \frac{f_2(n)}{n!(16ax)^n} \right] \quad (27)$$

$$f_1(n) = (2n-1)^2 (2n-3)^2 \ldots 1 \quad (28)$$

$$f_2(n) = (-1)^n (4 - (2n-1)^2) \times (4 - (2n-3)^2) \ldots (4 - 1). \quad (29)$$

The time integrals become then

$$\int_{t_1}^{t_2} T^* \, dt = \frac{1}{4\pi k\sqrt{(ad)}} \left[E_1(B) + \sum_{n=1}^{\infty} \frac{f_1(n) b^n}{n!(16a)^n} \Gamma(-n, B) \right] \quad (30)$$

$$\int_{t_1}^{t_2} q^* \, dt = \frac{K}{2\pi k d\sqrt{(ad)}} \left\{ \frac{1}{b} e^{-B} [(r - r_i) r_{,n} + (z - z_i) z_{,n}] + [rr_{,n} + (z - z_i)z_{,n}] \sum_{n=1}^{\infty} \frac{f_1(n) b^{n-1}}{n!(16a)^n} \Gamma(1-n, B) - r_i r_{,n} \sum_{n=1}^{\infty} \frac{f_2(n) b^{n-1}}{n!(16a)^n} \Gamma(1-n, B) \right\} \quad (31)$$

where $b = 1 - 2a$ and $B = bC$. The incomplete gamma functions can now be computed from $\Gamma(0, B)$ by using the recurrence relation [15, p. 262]

$$\Gamma(-n, B) = -\frac{1}{n} \left[\Gamma(1-n, B) - \frac{e^{-B}}{B^n} \right] \quad (32)$$

$$\Gamma(0, B) = E_1(B). \quad (33)$$

For computational purposes, the exponential integral can be computed (with sufficient accuracy) using the following polynomial and rational approximations [15, p. 231]

$$E_1(B) = -0.57721566 + 0.99999193B - 0.24991055B^2 + 0.05519968B^3 - 0.00976004B^4 + 0.00107857B^5 - \ln B \quad \text{for } 0 \leqslant B \leqslant 1 \quad (34)$$

$$E_1(B) = (B^4 + 8.57332874B^3 + 18.05901697B^2 + 8.63476089B + 0.26777373) / [(B^4 + 9.57332234B^3 + 25.63295614B^2 + 21.09965308B + 3.95849692) B e^B] \quad \text{for } 1 \leqslant B < \infty.$$

When the constant $a \to 0.5$ but x is small, we cannot use directly expansions (26) and (27). Alternatively, equation (17) may be written as

$$\int_{t_1}^{t_2} T^* \, dt = \frac{1}{2k\sqrt{(\pi d)}} \left[\int_C^{C'} I_0(2ax) x^{-1/2} e^{-x} \, dx + \int_{C'}^{\infty} I_0(2ax) x^{-1/2} e^{-x} \, dx \right] \quad (35)$$

where

$$\int_C^{C'} I_0(2ax) x^{-1/2} e^{-x} \, dx = \sum_{n=0}^{\infty} \frac{a^{2n}}{n!^2} [\Gamma(2n + \tfrac{1}{2}, C) - \Gamma(2n + \tfrac{1}{2}, C')] \quad (36)$$

and expansion (26) is now used to evaluate the second integral in (35). The same idea can be applied on calculating the flux time integral (21).

The discretized form of the boundary integral equation (16) is

$$\sum_{j=1}^{N} (H_{ij} - \delta_{ij} c_i) T_j = \sum_{j=1}^{N} G_{ij} Q_j - B_i \quad (37)$$

where the B_i term accounts for the initial conditions and

$$H_{ij} = \frac{k}{K}\int_{S_j} \phi^T \int_{t_1}^{t_2} q^* \,\mathrm{d}t r \,\mathrm{d}S;$$

$$G_{ij} = \frac{k}{K}\int_{S_j} \psi^T \int_{t_1}^{t_2} T^* \mathrm{d}t r \,\mathrm{d}S. \quad (38)$$

Each space integral in (38) can be evaluated using a standard Gaussian quadrature, except the ones in which the element j contains the node i, for in these cases the integrals become singular. A careful investigation on equation (30) shows that the singularity of G_{ii} is of the logarithmic type, thus integrable. Expanding the exponential integral, one can isolate the logarithmic term and integrate it analytically. All the remainder is non-singular and can be integrated by using a standard Gaussian quadrature.

The H_{ii} terms contain a logarithmic plus a $1/b$ singularity. The first one is directly integrable but the second is only integrable in the Cauchy principal value sense. Expanding the first term of each series in (31) in order to isolate the logarithmic singularity, we can evaluate both singular integrals analytically and all the remainder, which is non-singular, using a standard Gaussian quadrature. The expressions found are rather lengthy and are given explicitly in [17] for the case of constant and linear interpolation functions.

Two different time-marching schemes can be employed on the numerical solution of equation (16): the first treats each time step as a new problem and so, at the end of each step, temperature values at a sufficient number of internal points are calculated in order to be used as pseudo-initial values for the next step; in the other, the time integration process always starts at time t_0 and so, despite the increasing number of intermediate steps as the time progresses, temperature values at internal points do not need to be recomputed.

Although demanding a domain integral the former scheme has the advantage that if a constant time step is adopted, all matrices involved in equation (37) will also be constant throughout the analysis and so can be computed only once and stored. On the other hand, the latter scheme has the advantage that if T_0 satisfies Laplace's equation, the domain integral in (16) can be transformed into equivalent boundary integrals [17] and then a reduction of the dimensionality of the problem is effectively achieved. However, as the fundamental solution is dependent on the actual value of time, the matrices that appear in (37) have to be recomputed as the time progresses.

A comparison between the computer efficiency of both above-mentioned time-marching schemes carried out by the authors [18] showed that the first scheme is more economical than the second for general problems, although the second is more efficient when temperature values at only a few numbers of intermediate time steps are required, and also for problems involving domains extending to infinity. As these are not the cases of the examples analysed here, the first scheme was employed in the present work.

The domain is then discretized into cells and the integrals performed by using a numerical integration procedure, for instance, Hammer's quadrature rule. Assuming that the temperature values are computed directly at each integration point in the cells, the B_i terms in (37) are of the form

$$B_i = \sum_{m=1}^{M}\sum_{l=1}^{L} T^*_{il} r_l w_l J_l (T_l)_{t=t_1} = \sum_{m=1}^{M}\sum_{l=1}^{L} B_{il}(T_l)_{t=t_1}. \quad (39)$$

Alternatively, if we assume that the temperature inside each cell varies according to a certain interpolation function that should be of the same order as the one which prescribes the variation of temperature over the boundary elements, we have

$$B_i = \sum_{v=1}^{V}\sum_{m=1}^{M}\sum_{l=1}^{L} T^*_{il} r_l w_l J_l \gamma^T_{lv} (T_v)_{t=t_1} = \sum_{v=1}^{V} B_{iv}(R_v)_{t=t_1}. \quad (40)$$

A more detailed discussion on the computation of the domain integral can be found in [17].

APPLICATIONS

In order to show the numerical accuracy of the formulation developed in this paper, three different examples were analyzed, including a problem with time-dependent boundary conditions. Due to the symmetry with respect to the r-axis, only one half of the cross-section needed to be discretized in all problems studied. Symmetry is taken into account by a direct condensation process with integration over reflected elements, such that no discretization of the axis of symmetry is necessary. All the examples were studied using constant boundary elements with four Gaussian points and computing the domain integrals as in equation (39), using Hammer's quintic quadrature rule.

SOLID CYLINDER WITH CONVECTION

The first example analyzed was that of a solid cylinder at unit initial temperature, subjected to the following boundary conditions

$$T = 0 \quad \text{at} \quad r = a$$

$$q = 2T \quad \text{at} \quad z = \pm l.$$

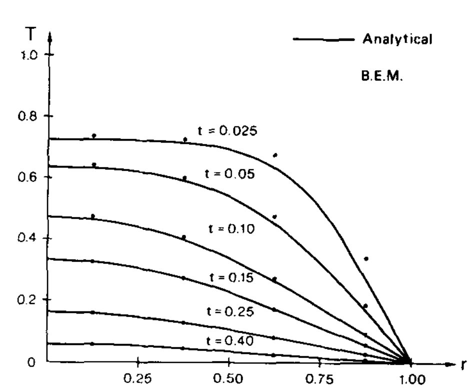

FIG. 2. Temperature at $z = \pm l$.

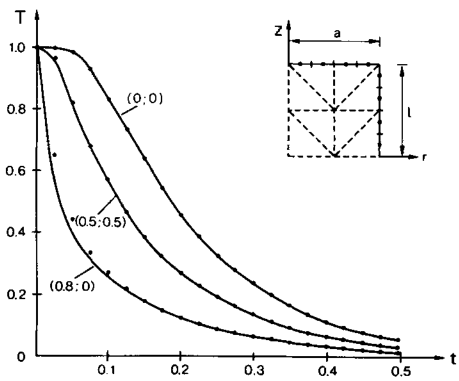

FIG. 3. Temperature at internal points.

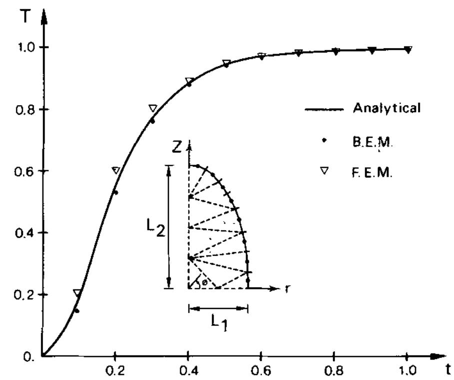

FIG. 4. Temperature at the centre point of a prolate spheroid.

The discretization adopted is shown in Fig. 3. The numerical values assumed for the cross-section were $a = 1$, $l = 1$ and for simplicity, the coefficients k and K were also assumed to be unity.

Results are compared in Figs. 2 and 3 against an available analytical solution [13], showing good agreement. The analysis was performed with a time step $\Delta t = 0.025$ and took about 4 s of CPU time in an IBM 360/195 computer to converge to a steady-state (20 time intervals).

PROLATE SPHEROIDAL SOLID

This example studies a prolate spheroid initially at zero temperature and subjected to a unit surface temperature at $t = 0$. A parametric representation of points on its surface, in the rz plane, may be written as

$$r = L_1 \cos\phi$$
$$z = L_2 \sin\phi$$

where the ϕ angle is indicated in Fig. 4.

The discretization employed is shown in the figure and the numerical values assumed for this analysis were $K = k = 1$, $L_2 = 2$. Results for the centre point ($r = z = 0$) are compared in Fig. 4 against an analytical solution [19] and a finite element solution [20] obtained with parabolic three-dimensional isoparametric elements. The finite element analysis was performed with a $\Delta t = 0.025$ whereas the boundary element solution employed a $\Delta t = 0.050$. The total CPU time was 4.5 s, for 20 time steps.

SPHERE WITH THERMAL SHOCK

An alternative method of analysing transient problems using boundary elements is in conjunction with Laplace transforms [21]. However, this method does not produce good results for problems with time-dependent boundary conditions, due to numerical problems in the inverse transformation process [22]. The step-by-step method shown in this paper does not present this drawback.

To show the accuracy of the method for problems with time-dependent boundary conditions, a sphere subjected to thermal shocks was analysed. The discretization of the sphere is shown in Fig. 5, together with the variation with respect to time of the boundary temperature. It consists of two thermal shocks, one of which is imposed at $t = 0$, the other at time t_0.

Results are presented in Figs. 6–8 for different values of t_0 and compared against analytical solutions [13]. The accuracy of the boundary element solutions is very good for all values of t_0. Each analysis took about 3 s of CPU computer-time to run.

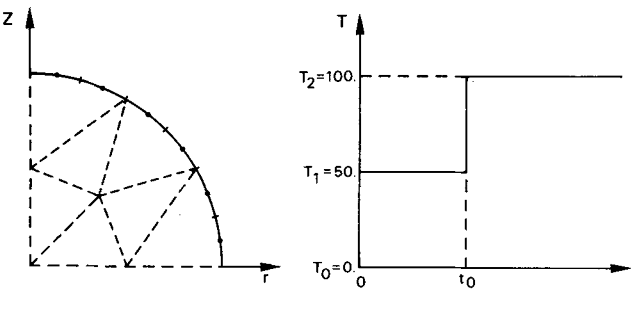

FIG. 5. Discretization and variation of boundary temperature on time for sphere.

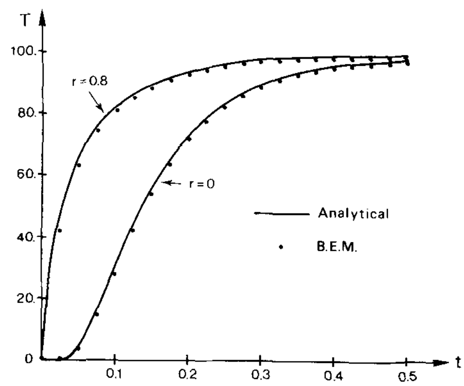

FIG. 6. Temperature at internal points for thermal shock at $t = 0$.

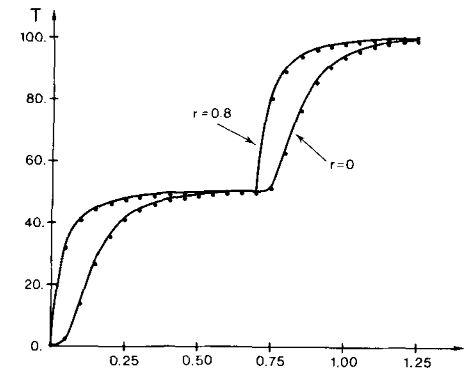

FIG. 8. Temperature at internal points for thermal shocks at $t = 0$ and $t = 0.7$.

CONCLUSIONS

The boundary elements formulation for axisymmetric transient heat conduction as presented in this paper proved to be a powerful numerical tool for many practical problems, including problems with time-dependent boundary conditions. Despite the fact that the influence coefficients on the system matrices are evaluated using series expansions, all the series that appear in the formulation converge very quickly and so the computing effort involved is not great, as shown by the small computer CPU times required to run the examples.

The use of a fundamental solution which is space- and time-dependent eliminates the need of integrating step-by-step on time using a finite difference-type scheme. But in order to compute the time integrals analytically, time-stepping is also necessary to obtain accurate results although the steps can be comparatively large. The assumption of constant variation on time for the variables, as was done in this paper, is not restrictive and higher order interpolation functions can be introduced. In fact, the use of linear time interpolation functions still permits the analytical evaluation of the time integrals and allows even larger time steps to be employed. Results on these developments will be published in a future paper.

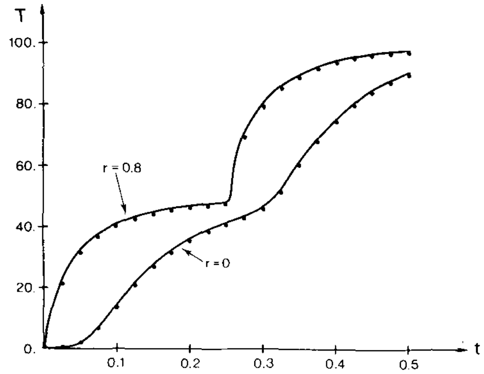

FIG. 7. Temperature at internal points for thermal shocks at $t = 0$ and $t = 0.25$.

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UNE FORMULATION DE LA METHODE DES ELEMENTS LIMITES POUR LA CONDUCTION THERMIQUE AXISYMETRIQUE ET VARIABLE

Résumé — On développe la formulation de la méthode des éléments limites pour l'analyse des problèmes de conduction thermique axisymétrique et variable. La solution fondamentale axisymétrique, dépendante du temps, est obtenue par intégration directe d'une solution tridimensionnelle. Du fait de la complexité, les développements en série sont introduits de façon à faciliter l'évaluation analytique des intégrales de temps qui apparaissent dans la formulation. On présente plusieurs résultats numériques, en incluant des problèmes avec des conditions aux limites dépendant du temps, et il est montré la possibilité d'utiliser les éléments limites dans l'espace et dans le temps pour résoudre des problèmes de conduction thermique axisymétriques.

EINE FORMULIERUNG DER RANDELEMENTMETHODE FÜR ACHSENSYMMETRISCHE INSTATIONÄRE WÄRMELEITUNG

Zusammenfassung — In der vorliegenden Arbeit wird eine Formulierung der Randelementmethode zur Analyse von achsensymmetrischen, instationären Wärmeleitproblemen entwickelt. Die achsensymmetrische zeitabhängige allgemeine Lösung wird durch direkte Integration der dreidimensionalen Lösung erhalten. Infolge ihrer Komplexität mußten Reihenentwicklungen eingeführt werden, um die analytische Berechnung der Zeitintegrale zu ermöglichen, die in der Formulierung auftreten. Mehrere Ergebnisse einer numerischen Analyse werden angegeben einschließlich von Problemen mit zeitabhängigen Randbedingungen. Sie demonstrieren die Möglichkeit, Randelemente in Raum und Zeit zu benutzen, um achsensymmetrische Wärmeleitprobleme zu lösen.

ФОРМУЛИРОВКА МЕТОДА ГРАНИЧНОГО ЭЛЕМЕНТА ДЛЯ ОСЕСИММЕТРИЧНЫХ НЕСТАЦИОНАРНЫХ ЗАДАЧ ТЕПЛОПРОВОДНОСТИ

Аннотация — В настоящей работе разрабатывается метод граничного элемента для анализа осесимметричных нестационарных задач теплопроводности. Осесимметричное фундаментальное решение, зависящее от времени, получено путем прямого интегрирования трехмерной задачи. В виду сложности полученного решения были применены разложения в ряд, что позволило осуществить аналитическую оценку интегралов, зависящих от времени. Представлены результаты численного анализа, в том числе для задач с граничными условиями, зависящими от времени. Результаты указывают на возможность использования граничных элементов в координатах и времени для решения осесимметричных задач теплопроводности.